NEW PROOF ON SOME SHARP DOUBLE INTEGRAL INEQUALITIES OF THE HERMITE-HADAMARD TYPE

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Abstract. In this paper, we derive a new proof on some sharp double integral inequalities of the Hermite-Hadamard type. Our approach is mainly based on well-known Taylor's theorem with the integral remainder.

1. introduction

Let $f(x)$ be a convex function on the closed interval $[a,b]$, the well-known Hermite-Hadamard's inequality can be expressed as ([2]):

$$f\left(\frac{a+b}{2}\right) \leqq \frac{1}{b-a}\int\limits_a^b f(x)\,\mathrm{d}x \leqq \frac{f(a)+f(b)}{2}. \tag{1}$$

It is well known that Hermite-Hadamard's inequality is an important cornerstone in mathematical analysis and optimization. There is a growing literature considering its refinements and interpolations now. Recently, Ujević obtained the following similar inequalities for convex functions

$$\frac{f(a)+f(b)}{2} - \frac{1}{8}S \leqq \frac{1}{b-a}\int\limits_a^b f(x)\,\mathrm{d}x \leqq f\left(\frac{a+b}{2}\right) + \frac{1}{8}S, \tag{2}$$

where $S = (f'(b) - f'(a))(b-a)$.

In this paper, we shall prove the following sharp double integral inequalities of the Hermite-Hadamard type.

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I, a < b$. If $f : I \to \mathbb{R}$ is differentiable, $m = \inf\limits_{x\in[a,b]} f''(x)$ and $M = \sup\limits_{x\in[a,b]} f''(x)$. Then we have*

$$f\left(\frac{a+b}{2}\right) + \frac{m}{24}(b-a)^2 \leqq \frac{1}{b-a}\int\limits_a^b f(x)\,\mathrm{d}x \leqq \frac{f(a)+f(b)}{2} - \frac{m}{12}(b-a)^2 \tag{3}$$

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and

$$\frac{f(a)+f(b)}{2}-\frac{M}{12}(b-a)^2 \leqq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqq f\left(\frac{a+b}{2}\right)+\frac{M}{24}(b-a)^2. \quad (4)$$

The inequalities (3) are sharp in the sense that the constants $\frac{1}{24}$ in the left-hand and $\frac{1}{12}$ in the right-hand cannot be replaced by a larger one, respectively. The inequalities (4) are sharp in the sense that the constants $\frac{1}{12}$ in the left-hand and $\frac{1}{24}$ in the right-hand cannot be replaced by a smaller one, respectively.

Remark 1. (1) *If $f'' \geqq 0$, $t \in [a,b]$, i.e. f is a convex function, thus we can set $m = 0$ in (3). Then, we recapture the well-known Hermite-Hadamard inequalities for convex functions.*

(2) *If $f'' \leqq 0$, $t \in [a,b]$, i.e. f is a concave function, thus we can set $M = 0$ in (4). Then, we get the following inequalities for concave functions.*

$$\frac{f(a)+f(b)}{2} \leqq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqq f\left(\frac{a+b}{2}\right). \quad (5)$$

Remark 2. *We also note that inequalities* (3) *and* (4) *have been proved in* [3]*. In this short note, we shall use an other approach.*

Before proving our main theorem, we need an essential lemma below. It is well-known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 1 (See [1], Theorem 1)**.** *Let $f : [a,b] \to \mathbb{R}$ and let r be a positive integer. If f is such that $f^{(r-1)}$ is absolutely continuous on $[a,b]$, $x_0 \in (a,b)$ then for all $x \in (a,b)$ we have*

$$f(x) = T_{r-1}(f, x_0, x) + R_{r-1}(f, x_0, x)$$

where $T_{r-1}(f, x_0, \cdot)$ is Taylor's polynomial of degree $r-1$, that is,

$$T_{r-1}(f, x_0, x) = \sum_{k=0}^{r-1} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$$

and the remainder can be given by

$$R_{r-1}(f, x_0, x) = \int_{x_0}^{x} \frac{(x-t)^{r-1} f^{(r)}(t)}{(r-1)!}\,\mathrm{d}t \quad (6)$$

By a simple calculation, the remainder in (6) can be rewritten as

$$R_{r-1}(f, x_0, x) = \int_0^{x-x_0} \frac{(x-x_0-t)^{r-1} f^{(r)}(x_0+t)}{(r-1)!}\,\mathrm{d}t$$

which helps us to deduce a similar representation of f as following

$$f\left(x+u\right)=\sum_{k=0}^{r-1}\frac{u^{k}}{k!}f^{(k)}\left(x\right)+\int\limits_{0}^{u}\frac{\left(u-t\right)^{r-1}}{\left(r-1\right)!}f^{(r)}\left(x+t\right)\mathrm{d}t. \tag{7}$$

2. Proofs of Theorem 1

Let

$$F\left(x\right)=\int\limits_{a}^{x}f\left(t\right)\mathrm{d}t.$$

Then

$$F\left(b\right)=F\left(\frac{a+b}{2}\right)+\frac{b-a}{2}F'\left(\frac{a+b}{2}\right)+\int\limits_{0}^{\frac{b-a}{2}}\left(\frac{b-a}{2}-t\right)F''\left(\frac{a+b}{2}+t\right)\mathrm{d}t.$$

Equivalently,

$$F\left(b\right)=F\left(\frac{a+b}{2}\right)+\frac{b-a}{2}f\left(\frac{a+b}{2}\right)+\int\limits_{0}^{\frac{b-a}{2}}\left(\frac{b-a}{2}-t\right)f'\left(\frac{a+b}{2}+t\right)\mathrm{d}t.$$

Similarly,

$$\begin{aligned}F\left(a\right)=&F\left(\frac{a+b}{2}\right)+\frac{a-b}{2}f\left(\frac{a+b}{2}\right)+\int\limits_{0}^{\frac{a-b}{2}}\left(\frac{a-b}{2}-t\right)f'\left(\frac{a+b}{2}-t\right)\mathrm{d}t\\ \overset{t:=-t}{=}&F\left(\frac{a+b}{2}\right)-\frac{b-a}{2}f\left(\frac{a+b}{2}\right)+\int\limits_{0}^{\frac{b-a}{2}}\left(\frac{b-a}{2}-t\right)f'\left(\frac{a+b}{2}-t\right)\mathrm{d}t.\end{aligned}$$

Therefore,

$$\begin{aligned}\int\limits_{a}^{b}f\left(x\right)\mathrm{d}x-\left(b-a\right)f\left(\frac{a+b}{2}\right)&=F\left(b\right)-F\left(a\right)-\left(b-a\right)f\left(\frac{a+b}{2}\right)\\ &=\int\limits_{0}^{\frac{b-a}{2}}\left(\frac{b-a}{2}-t\right)\left(f'\left(\frac{a+b}{2}+t\right)-f'\left(\frac{a+b}{2}-t\right)\right)\mathrm{d}t\\ &\geqq\int\limits_{0}^{\frac{b-a}{2}}\left(\frac{b-a}{2}-t\right)2tm\,\mathrm{d}t\\ &=\frac{m}{24}\left(b-a\right)^{3}.\end{aligned}$$

On the other hand,

$$\int\limits_{a}^{b}f\left(x\right)\mathrm{d}x=F\left(b\right)-F\left(a\right)$$

$$= (b-a)\,F'(a) + \int_0^{b-a} (b-a-t)\,F''(a+t)\,\mathrm{d}t$$

and

$$\frac{b-a}{2}\left(f(a)+f(b)\right) = \frac{b-a}{2}\left(2f(a) + \int_0^{b-a} f'(a+t)\,\mathrm{d}t\right)$$

which helps us to deduce that

$$\begin{aligned}
\frac{b-a}{2}&\left(f(a)+f(b)\right) - \int_a^b f(x)\,\mathrm{d}x \\
&= \frac{b-a}{2}\int_0^{b-a} f'(a+t)\,\mathrm{d}t - \int_0^{b-a}(b-a-t)\,f'(a+t)\,\mathrm{d}t \\
&= \int_0^{b-a}\left(t-\frac{b-a}{2}\right) f'(a+t)\,\mathrm{d}t \\
&= \int_{\frac{b-a}{2}}^{b-a}\left(t-\frac{b-a}{2}\right) f'(a+t)\,\mathrm{d}t - \int_0^{\frac{b-a}{2}}\left(\frac{b-a}{2}-t\right) f'(a+t)\,\mathrm{d}t \\
&= \int_0^{\frac{b-a}{2}}\left(\frac{b-a}{2}-t\right) f'(b-t)\,\mathrm{d}t - \int_0^{\frac{b-a}{2}}\left(\frac{b-a}{2}-t\right) f'(a+t)\,\mathrm{d}t \\
&= \int_0^{\frac{b-a}{2}}\left(\frac{b-a}{2}-t\right)\left(f'(b-t)-f'(a+t)\right)\mathrm{d}t \\
&\geqq \int_0^{\frac{b-a}{2}}\left(\frac{b-a}{2}-t\right)(b-a-2t)\,m\,\mathrm{d}t \\
&= \frac{m}{12}(b-a)^3.
\end{aligned}$$

If we now substitute $f(x) = (x-a)^2$ in the inequalities then we find that the left-hand side, middle term and right-hand side are all equal to $\frac{(b-a)^2}{3}$. Thus, the inequalities (3) are sharp in the usual sense.

The proof of (3) is completed. The proof of (4) is similar.

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